MATHEMATICS IN CONTRACT THEORY (PRELIMINARY VERSION)

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For general introduction to contract theory, refer to [Bolton and Dewatripont, 2005]. For mathematical theory for continuous-time models, refer to [Cvitanić and Zhang, 2013].

1. One-period model

1.1. **Problem formulation.** To understand the flavor of questions in contract theory, we start with an one-period model in this section.

Consider an agent whose utility is modelled by an exponential utility with the risk aversion γ_A, i.e., $U_A(c) = -\frac{1}{\gamma_A}e^{-\gamma_A c}$. If hired at time 0, this agent manages a project, which produces an output at time 1. The output is random, and it is modelled by a normal random variable X with the mean α and the standard deviation σ, i.e., $X \sim N(\alpha, \sigma)$. The mean α is determined by the agent's *effort*. If the agent works hard, α is high, otherwise α is low. The randomness of the output is characterized by σ which *cannot* be controlled by the agent. The effort is costly to the agent. The cost is given by a cost function $g(\alpha)$ which is the monetary cost to the agent if he exerts effort α. We consider the following cost function

$$g(\alpha) = \tfrac{1}{2}\alpha^2.$$

If hired, the agent will receive a lump-sum compensation ξ at time 1. We call this compensation the contract payment, or simply contract.

The agent chooses his optimal strategy to maximize the expected effort net cost, i.e.,

$$\mathbb{E}\big[U_A(\xi - \tfrac{1}{2}\alpha^2)\big] \to \text{Max}.$$

Assuming that this optimization problem has an optimizer α^ξ, $\mathbb{E}[U_A(\xi - \frac{1}{2}(\alpha^\xi)^2)]$ is the optimal value associated to the contract ξ for the agent. The agent has a reservation value $U_A(R)$, which describes the minimal utility that the agent needs to take on a job; i.e., the agent will only take on a contract if the associated optimal value is not smaller than his reservation value.

Consider a principal whose utility is modelled by an exponential utility with the risk aversion γ_P, i.e., $U_P(c) = -\frac{1}{\gamma_P}e^{-\gamma_P c}$. The principal wants to hire the agent to work on the project. However, the principal cannot monitor or observe agent's effort. The principal can only observe the realization of X at time 1. (It is impossible to estimate the mean

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of a normal random variable with only one realization.) Therefore, there is information asymmetry between the agent and the principal. This information asymmetry is called *moral hazard* in the economics literature.

At time 1, the principal receives the output X and pays the agent ξ. The principal aims to choose contract ξ to maximize her expected utility, i.e.,

$$\mathbb{E}\big[U_P(X^\xi - \xi)\big] \to \text{Max},$$

subject to

$$\mathbb{E}\big[U_A(\xi - \tfrac{1}{2}(\alpha^\xi)^2)\big] \geq U_A(R).$$

Here X^ξ is the output when the agent employs the effort α^ξ. The constraint is called agent's *participation constraint*. The goal of this problem is to find principal's optimal contract ξ and agent's optimal effort a^ξ.

1.2. **Agent's optimization problem.** First, it is not optimal for the principal to pay a constant salary ξ, because if agent receives a compensation which does not depend on the output, he will not exert any costly effort at all. Therefore, ξ needs to be random and depends on X. Let us consider the following linear contract

$$\xi = aX + b, \quad a, b \in \mathbb{R}, \tag{1.1}$$

where b is the fixed salary and aX is the performance-based compensation. The constants a and b are called *contract sensitivities*.

For a given linear contract in (1.1), the agent wants to maximize the expected utility

$$\mathbb{E}\Big[-\frac{1}{\gamma_A} e^{-\gamma_A(aX+b-\frac{1}{2}\alpha^2)}\Big].$$

Recall that

$$\mathbb{E}[e^{cX}] = e^{\alpha c + \frac{1}{2}\sigma^2 c^2}. \tag{1.2}$$

The previous expected utility equals to

$$-\frac{1}{\gamma_A} e^{-\gamma_A b - \gamma_A a\alpha + \frac{1}{2}\sigma^2 a^2 \gamma_A^2 + \frac{1}{2}\gamma_A \alpha^2}.$$

Its optimizer minimizes $-\gamma_A a\alpha + \frac{1}{2}\gamma_A\alpha^2$. Therefore, the optimal effort is

$$a^\xi = a, \tag{1.3}$$

and agent's optimal value is

$$-\frac{1}{\gamma_A} e^{-\gamma_A b + \frac{1}{2}\sigma^2 a^2 \gamma_A^2 - \frac{1}{2}\gamma_A a^2}.$$

1.3. **Principal's optimization problem.** Given agent's optimal effort a^ξ and the associated output X^ξ, the principal aims to maximize

$$\begin{aligned} &\mathbb{E}\big[U_P(X^\xi - \xi)\big] \\ =& -\tfrac{1}{\gamma_P}\mathbb{E}\big[e^{-\gamma_P(X^\xi - aX^\xi - b)}\big] \\ =& -\tfrac{1}{\gamma_P} e^{\gamma_P b - \gamma_P(1-a)a + \frac{1}{2}\sigma^2\gamma_P^2(1-a)^2}, \end{aligned} \tag{1.4}$$

where (1.2) is used to obtain the second identity.

Observe that principal's optimal utility is decreasing in b. Therefore the principal wants to choose the smallest b such that the participation constraint is satisfied. This implies

$$-\tfrac{1}{\gamma_A} e^{-\gamma_A b - \gamma_A a\alpha + \frac{1}{2}\sigma^2 a^2\gamma_A^2 + \frac{1}{2}\gamma_A \alpha^2} = -\tfrac{1}{\gamma_A} e^{-\gamma_A R},$$

leading to $b = \frac{1}{2}\gamma_A\sigma^2 a^2 - \frac{1}{2}a^2 + R$.

Plugging the previous expression of b back to (1.4), we obtain

$$\mathbb{E}\big[U_P(X^\xi - \xi)\big] = -\tfrac{1}{\gamma_P}\exp\big(\gamma_P R + \tfrac{1}{2}\gamma_A\gamma_P\sigma^2 a^2 - \tfrac{1}{2}\gamma_P a^2 - \gamma_P(1-a)a + \tfrac{1}{2}\sigma^2\gamma_P^2(1-a)^2\big).$$

Therefore, we just need to minimize the exponent of the right-hand side. The optimizer is

$$a^* = \frac{1+\gamma_P\sigma^2}{1+(\gamma_A+\gamma_P)\sigma^2}.$$

Proposition 1.1. *Among all linear contracts of type* (1.1)*, the optimal linear contract is given by*

$$\xi = a^* X + b^*, \qquad \textit{where } a^* = \frac{1+\gamma_P\sigma^2}{1+(\gamma_A+\gamma_P)\sigma^2} \textit{ and } b^* = \tfrac{1}{2}\gamma_A\sigma^2(a^*)^2 - \tfrac{1}{2}(a^*)^2 + R.$$

Remark 1.2. From the expression of a^*, we can see that $a^* \to 1$, when either $\sigma \to 0$, $\gamma_A \to 0$, or $\gamma_P \to \infty$. Therefore, the principal sells the whole project to the agent for cash at time 0, when either the project is riskless, the agent is risk neutral, or the principal is extremely risk averse.

Remark 1.3. The optimal contract in the linear class is obtained in the previous proposition. However, if the principal is allowed to choose from nonlinear contracts, it is not clear whether nonlinear contracts will improve her utility. Consider a N-period model, where the output at the end of period N is $X_1 + X_2 + \cdots X_N$. If the principal can observe all X_i, she can choose from path-dependent contracts $f(X_1, X_2, \ldots, X_N)$. In this case, it is restrictive to consider only linear contracts. We will tackle these problems in a continuous-time framework in the next section.

2. Holmström and Milgrom (1987)

2.1. **Preliminaries.**

2.2. **Problem formulation.** We present [Holmstrom and Milgrom, 1987] in this section.

Let $(\Omega, \mathbb{F}, (\mathcal{F}_{t\in[0,T]}), \mathbb{P})$ be a filtered probability space, where the filtration is the augmented filtration generated by a 1-dimensional standard Brownian motion B.

The agent, if hired at time 0, works for a project continuously from time 0 to T. His effort is an adapted process α. The output process X satisfies the dynamics

$$dX_t = \alpha_t dt + \sigma dB_t^\alpha, \tag{2.1}$$

where σ is a positive constant,

$$B_t^\alpha = B_t - \int_0^t \tfrac{\alpha_s}{\sigma} ds. \tag{2.2}$$

One can observe that $X = \sigma B$. Define a new probability measure $\mathbb{P}^\alpha$ via

$$\tfrac{d\mathbb{P}^\alpha}{d\mathbb{P}} = \exp\big(\int_0^T \tfrac{\alpha_s}{\sigma} dB_s - \tfrac{1}{2}\int_0^T (\tfrac{\alpha_s}{\sigma})^2 ds\big).$$

We assume that α satisfies enough integrability so that the right-hand side has expectation 1 under $\mathbb{P}$. Therefore, $\mathbb{P}^\alpha$ is a probability measure equivalent to $\mathbb{P}$. In what follows, we denote $\mathbb{E}^{\mathbb{P}^\alpha}[\cdot]$ by $\mathbb{E}^\alpha$ and $\mathbb{E}^{\mathbb{P}^\alpha}[\cdot\,|\,\mathcal{F}_t]$ by $\mathbb{E}_t^\alpha[\cdot]$. By Girsanov theorem, B^α is a Brownian motion under $\mathbb{P}^\alpha$. Therefore, if α is deterministic, X_T has normal distribution under $\mathbb{P}^\alpha$ with the mean $\int_0^T \alpha_t dt$ and the standard deviation $\sigma\sqrt{T}$.

Remark 2.1. The formulation in (2.1) is called the *weak formulation.* In such a formulation, controlling the drift of X is equivalent to controlling the probability measure $\mathbb{P}^\alpha$.

The cost of agent's effort is assumed to be $\int_0^T \frac{1}{2}\alpha_s^2 ds$. The agent chooses his optimal strategy to maximize

$$\mathbb{E}\big[U_A(\xi - \int_0^T \tfrac{1}{2}\alpha_s^2 ds)\big].$$

Assume that this problem admits an optimal strategy α.

The principal observes the output process continuously. However, she cannot distinguish the drift α from the noise. In other words, when the output process increases, she does not know if the agent works hard or she is just lucky. This information asymmetry between the principal and agent creates moral hazard. Given agent's optimal effort a^ξ and the associated output process X^ξ, the principal aims to maximize

$$\mathbb{E}\big[U_P(X_T^\xi - \xi)\big],$$

subject to agent's participation constraint.

2.3. **Agent's optimization problem.** Let us first define a dynamic version of agent's value function:

$$v_t := \text{ess sup}_\alpha \mathbb{E}_t^\alpha\big[U_A(\xi - \int_t^T \tfrac{1}{2}\alpha_s^2 ds)\big]. \tag{2.3}$$

One can think v_t as the optimal value at time t if the agent acts optimally from time t onwards. Therefore v is called the *continuation value*. Define u via

$$U_A(u_t) = v_t. \tag{2.4}$$

The process u is called the *certainty equivalence*, which is the monetary equivalence of agent's optimal value.

Now we present a heuristic argument to identify the dynamics of u. Assume that u has the following dynamics

$$du_t = H_t dt + Z_t dB_t,$$

where H will be determined in what follows.

Since U_A is an exponential utility,

$$-\tfrac{1}{\gamma} e^{-\gamma_A u_t} e^{\gamma_A \int_0^t \frac{1}{2}\alpha_s^2 ds} = \text{ess sup}_\alpha \mathbb{E}_t^\alpha\big[-\tfrac{1}{\gamma_A}\exp\big(-\gamma_A\big(\xi - \int_0^T \tfrac{1}{2}\alpha_s^2 ds\big)\big)\big].$$

The *martingale principal* expects that

- $-\frac{1}{\gamma} e^{-\gamma_A u_t} e^{\gamma_A \int_0^t \frac{1}{2}\alpha_s^2 ds}$ is a supermartingale under $\mathbb{P}^\alpha$ for any strategy α;
- $-\frac{1}{\gamma} e^{-\gamma_A u_t} e^{\gamma_A \int_0^t \frac{1}{2}\alpha_s^2 ds}$ is a martingale under $\mathbb{P}^{\alpha^*}$ for the optimal strategy α^*.

We will use this martingale principal to identify H, hence the dynamics of u. To this end, applying Itô's formula, we obtain

$$\begin{aligned}\frac{d - \frac{1}{\gamma} e^{-\gamma_A u_t} e^{\gamma_A \int_0^t \frac{1}{2}\alpha_s^2 ds}}{-e^{-\gamma_A u_t} e^{\gamma_A \int_0^t \frac{1}{2}\alpha_s^2 ds}} =& \big[-H_t + \tfrac{1}{2}\gamma_A Z_t^2 + \tfrac{1}{2}\alpha_t^2\big]dt - Z_t dB_t \\ =& \big[-H_t + \tfrac{1}{2}\gamma_A Z_t^2 + \tfrac{1}{2}\alpha_t^2 - Z_t\tfrac{\alpha_t}{\sigma}\big]dt - Z_t dB_t^\alpha,\end{aligned}$$

where the second identity follows from (2.2). Having $-\frac{1}{\gamma} e^{-\gamma_A u_t} e^{\gamma_A \int_0^t \frac{1}{2}\alpha_s^2 ds}$ as a supermartingale implies that

$$-H + \tfrac{1}{2}\gamma_A Z^2 + \tfrac{1}{2}\alpha^2 - Z\tfrac{\alpha}{\sigma} \geq 0 \quad dt \times \Omega\, a.s.$$

Moreover, having one α^* such that $-\frac{1}{\gamma} e^{-\gamma_A u_t} e^{\gamma_A \int_0^t \frac{1}{2}(\alpha_s^*)^2 ds}$ is a martingale implies that the previous inequality is an identity when $\alpha = \alpha^*$. Therefore,

$$\begin{aligned}H =& \inf_\alpha \big\{\tfrac{1}{2}\alpha^2 - Z\tfrac{\alpha}{\sigma}\big\} + \tfrac{1}{2}\gamma_A Z^2 \\ =& \tfrac{1}{2}(\gamma_A - \tfrac{1}{\sigma^2})Z^2,\end{aligned}$$

and the optimal α is

$$\alpha^* = \tfrac{Z}{\sigma}.$$

Therefore, the martingale principal implies that

$$du_t = \tfrac{1}{2}(\gamma_A - \tfrac{1}{\sigma^2})Z_t^2 dt + Z_t dB_t, \quad u_T = \xi, \tag{2.5}$$

where the terminal condition follows from (2.3) and (2.4).

One can view (2.5) as a Backward Stochastic Differential Equation (BSDE), whose solution is a pair of adapted processes (u, Z). The BSDE (2.5) can be solved explicitly. Consider the process $e^{(\frac{1}{\sigma^2}-\gamma_A)u_t}$. Itô's formula implies that

$$de^{(\frac{1}{\sigma^2}-\gamma_A)u_t} = \big(\tfrac{1}{\sigma^2} - \gamma_A\big)e^{(\frac{1}{\sigma^2}-\gamma_A)u_t} Z_t dB_t, \quad e^{(\frac{1}{\sigma^2}-\gamma_A)u_T} = e^{(\frac{1}{\sigma^2}-\gamma_A)\xi}.$$

Given that $\mathbb{E}\big[e^{(\frac{1}{\sigma^2}-\gamma_A)\xi}\big] < \infty$, we have

$$e^{(\frac{1}{\sigma^2}-\gamma_A)u_t} = \mathbb{E}_t\big[e^{(\frac{1}{\sigma^2}-\gamma_A)\xi}\big],$$

and Z is identified by the martingale representation theorem. In conclusion, for any contract ξ such that $\mathbb{E}\big[e^{(\frac{1}{\sigma^2}-\gamma_A)\xi}\big] < \infty$, we can identify (u, Z) satisfying (2.5). This pair is also unique in the class where $e^{(\frac{1}{\sigma^2}-\gamma_A)u}$ is of class (D).

The equation (2.5) also provides a handy representation of the contract ξ. Given u_0 and the process Z, running the dynamics (2.5) forward in time yields

$$\xi = u_0 + \int_0^T \tfrac{1}{2}(\gamma_A - \tfrac{1}{\sigma^2})Z_t^2 dt + \int_0^T Z_t dB_t. \tag{2.6}$$

This representation will be useful for principal's optimization problem.

We have derived the dynamics of u from the martingale principal. Let us now start from (2.5) and verify that its solution is the certainty equivalence defined in (2.4).

Proposition 2.2. *We restrict the contract ξ and agent's strategy α to the class such that the BSDE* (2.5) *admits a solution* (u, Z) *and* $\exp\big(-\gamma_A(u - \int_0^\cdot \frac{1}{2}\alpha_s^2 ds)\big)$ *is of class (D). Then agent's optimal strategy is $\alpha^* = Z/\sigma$ and u is agent's certainty equivalence.*

Proof. For arbitrary α, the construction of H ensures that $-\frac{1}{\gamma_A}e^{-\gamma_A(u_t - \int_0^\cdot \frac{1}{2}\alpha_s^2 ds)}$ is a local supermartingale. Since we assume that this process if of class (D), it is also a supermartingale. Therefore

$$U_A(u_t) = -\tfrac{1}{\gamma_A}e^{-\gamma_A u_t} \geq \mathbb{E}_t^\alpha\Big[-\tfrac{1}{\gamma_A}e^{-\gamma_A\left(u_T - \int_t^T \frac{1}{2}\alpha_s^2 ds\right)}\Big] = \mathbb{E}_t^\alpha\Big[-\tfrac{1}{\gamma_A}e^{-\gamma_A\left(\xi - \int_t^T \frac{1}{2}\alpha_s^2 ds\right)}\Big].$$

When $\alpha = \alpha^*$, $-\frac{1}{\gamma_A}e^{-\gamma_A(u_t - \int_0^\cdot \frac{1}{2}(\alpha^*)_s^2 ds)}$ is a local martingale, hence a martingale due to its class (D) property. Therefore, the previous inequality is an identity, verifying the optimality of α^*. $\square$

2.4. **Principal's optimization problem.** Using (2.6), principal's expected utility is reduced to

$$
\begin{aligned}
&\mathbb{E}^{\alpha^*}\big[U_P(X_T^\xi-\xi)\big]\\
&=-\tfrac{1}{\gamma_P}\mathbb{E}^{\alpha^*}\Big[\exp\Big(-\gamma_P\Big(\int_0^T\tfrac{Z_t}{\sigma}dt+\int_0^T\sigma dB_t^{\alpha^*}-u_0-\int_0^T\tfrac{1}{2}(\gamma_A-\tfrac{1}{\sigma^2})Z_t^2dt-\int_0^T Z_tdB_t\Big)\Big)\Big]\\
&=-\tfrac{1}{\gamma_P}\mathbb{E}^{\alpha^*}\Big[\exp\Big(-\gamma_P\Big(-u_0+\int_0^T\big(\tfrac{Z_t}{\sigma}-\tfrac{1}{2}(\gamma_A+\tfrac{1}{\sigma^2})Z_t^2\big)dt+\int_0^T(\sigma-Z_t)dB_t^{\alpha^*}\Big)\Big)\Big],
\end{aligned}
$$

where (2.2) is used to obtain the second identity. Rewrite the right-hand side using the stochastic exponential

$$
\mathcal{E}\Big(\int_0^T-\gamma_P(\sigma-Z_t)dB_t^{\alpha^*}\Big)=\exp\Big(-\gamma_P\int_0^T(\sigma-Z_t)dB_t^{\alpha^*}-\tfrac{1}{2}\gamma_P^2\int_0^T(\sigma-Z_t)^2dt\Big).
$$

We obtain

$$
\begin{aligned}
&\mathbb{E}^{\alpha^*}\big[U_P(X_T^\xi-\xi)\big]\\
&=-\tfrac{1}{\gamma_P}\mathbb{E}\Big[\exp\Big(\gamma_Pu_0+\int_0^T f(Z_t)dt\Big)\mathcal{E}\Big(\int_0^T-\gamma_P(\sigma-Z_t)dB_t^{\alpha^*}\Big)\Big],
\end{aligned}
\tag{2.7}
$$

where

$$
f(Z)=\tfrac{1}{2}\gamma_P^2(\sigma-Z)^2-\gamma_P\tfrac{Z}{\sigma}+\tfrac{1}{2}\gamma_P\big(\gamma_A+\tfrac{1}{\sigma^2}\big)Z^2.
$$

Set $Z^*=\operatorname{argmin}f(Z)$. Calculation yields

$$
Z^*=\sigma\frac{1+\sigma^2\gamma_P}{1+\sigma^2(\gamma_A+\gamma_P)}.
$$

Now we are ready to present the optimal contract for the principal. In contrast to the one-period model, a linear contract is proved to be optimal in a large class of contracts, which satisfies proper integrability conditions and includes nonlinear and path-dependent contracts.

Theorem 2.3. *Consider the class of contracts specified in Proposition 2.2, moreover $\mathcal{E}\big(\int-\gamma_p(\sigma-Z_t)dB^{Z/\sigma}\big)$ is a $\mathbb{P}^{Z/\sigma}$-martingale. Then the linear contract*

$$
\xi=\frac{1+\sigma^2\gamma_P}{1+\sigma^2(\gamma_A+\gamma_P)}X_T+b,
$$

where $b=R+\frac{1}{2}(\gamma_A-\frac{1}{\sigma^2})(Z^*)^2T$, is the optimal contract.

Proof. From (2.7), we can see that principal's expected utility is a decreasing function of u_0. On the other hand, u_0 is agent's certainty equivalence at time 0. Therefore, in order to satisfy the participation constraint, the principal chooses $u_0 = R$. From (2.7), we also obtain

$$\begin{aligned}
&\mathbb{E}^{\alpha^*}\big[U_P(X_T^\xi - \xi)\big] \\
&\leq -\tfrac{1}{\gamma_P} e^{\gamma_P R + f(Z^*)T} \mathbb{E}^{\alpha^*}\Big[\mathcal{E}\Big(\int_0^T -\gamma_P(\sigma - Z_t) dB_t^{\alpha^*}\Big)\Big] \\
&= -\tfrac{1}{\gamma_P} e^{\gamma_P R + f(Z^*)T},
\end{aligned}$$

where the identity follows the assumption that the expectation of the stochastic exponential is 1. The inequality above is an identity, when Z in (2.6) is chosen as Z^*. Recalling that $B = X/\sigma$, we have verified the optimality of the contract in the statement. □

3. Sannikov (2008)

3.1. **Problem formulation.** Rather than a contract of lump sum payments in the last section, we consider a model of continuous payments in [Sannikov, 2008]. The output process X is formulated weakly as in (2.1), whose drift is controlled by the agent. The agent's preference is represented by a utility function U and the rate of cost for agent's effort is given by a function g. We do not specify explicitly forms for U and g, only assume that g is strictly convex. Rather than the monetary cost as in the last section, we consider $g(\alpha)$ as the cost to agent's utility per unit of time.

A contract is a stream of payments $\{c_t; t \geq 0\}$. The agent receives the cash flow c and consume it right away without saving. The agent's optimization problem is

$$r\mathbb{E}^\alpha\Big[\int_0^\infty e^{-rt}\big(u(c_t) - g(\alpha_t)\big)dt\Big] \to \text{Max},$$

where r is the discounting rate and the r in front of the expectation is a normalization constant. We assume that agent's optimization problem admits an optimal strategy α^*. Agent's reservation utility is R; i.e., the agent will not work for the project if the optimal value is less than R. Define agent's continuation value as

$$W_t = \text{ess sup}_\alpha r\mathbb{E}_t^\alpha\Big[\int_t^\infty e^{-r(s-t)}\big(u(c_s) - g(\alpha_s)\big)ds\Big]. \tag{3.1}$$

The agent has limited liability. He does not accept negative W in any situation.

The principal is risk neutral. She chooses the compensation stream c to maximize her expected profit

$$\sup_{c,\tau_e} r\mathbb{E}^{\alpha^*}\Big[\int_0^{\tau_0\wedge\tau_e} e^{-rt}\big(\alpha_t^* - c_t\big)dt\Big],$$

where τ_0 is the first time that agent's contintutation value reaches zero, i.e., $\tau_0 = \inf\{t \geq 0 : W_t = 0\}$, and τ_e. At time τ_0, the contract is terminated, the agent is no longer paid and he exerts zero effort after τ_0. At time τ_e, the principal retires the agent by paying him a constant rate forever, then agent stops working. This constant retirement rate is determined so that agent's continuation value remains the same at retirement time.

3.2. **Agent's optimization problem.** Let us first use the martingale principle to heuristically derive the dynamics of agent's continuation value W. Define $\tilde{W}$ as

$$\tilde{W} = e^{-rt}W_t + r\int_0^t e^{-rs}\big(U(c_s) - g(\alpha_s)\big)ds.$$

Given c, we expect from the martingale principle that $\tilde{W}$ is a supermartingale under $\mathbb{P}^\alpha$ for arbitrary strategy α, and a martingale under $\mathbb{P}^{\alpha^*}$ for the optimal α^*. Suppose that W has the dynamics

$$dW_t = H_t dt + r\sigma Z_t dB_t.$$

Itô's formula implies that

$$\begin{aligned} d\tilde{W}_t =& e^{-rt}\Big(H_t dt + r\sigma Z_t dB_t - rW_t dt + r\big(U(c_t) - g(\alpha_t)\big)dt\Big) \\ =& re^{-rt}\big(\tfrac{1}{r}H_t - W_t + \alpha_t Z_t + U(c_t) - g(\alpha_t)\big)dt + re^{-rt}\sigma Z_t dB_t^\alpha. \end{aligned}$$

The martingale principle implies that the drift is nonnegative for arbitrary α and zero for the optimal α^*. Therefore,

$$H = r\big(\inf_\alpha\{g(\alpha) - \alpha Z\} + W - U(c)\big) = r\big(-\hat{g}(Z) + W - U(c)\big),$$

where $\hat{g}(Z) = -\inf_\alpha\{g(\alpha) - \alpha Z\}$ and the optimal strategy $\alpha^* = (g')^{-1}(Z)$. As a result, W follows the dynamics

$$dW_t = r\big(-\hat{g}(Z_t) + W_t - U(c_t)\big)dt + r\sigma Z_t dB_t. \tag{3.2}$$

Rather than a terminal condition at a finite time, the terminal condition for (3.2) is replaced by the following *transversality condition*

$$\lim_{\tau\to\infty} \mathbb{E}^\alpha\big[e^{-r\tau}W_\tau\big] = 0, \quad \text{for strategy } \alpha \text{ and any stopping time } \tau \to \infty. \tag{3.3}$$

From a BSDE point of view, (3.2), together with (3.3), is called an infinite horizon BSDE, studied by Royer (2004).

The following result makes the heuristic argument rigorous. Let us restrict the compensation stream and agent's strategy α such that

$$\mathbb{E}^\alpha\Big[\int_0^\infty e^{-rt}U(c_t)dt\Big] < \infty \quad \text{and} \quad \mathbb{E}^\alpha\Big[\int_0^\infty e^{-rt}g(\alpha_t)dt\Big] < \infty.$$

Proposition 3.1. *For a given compensation stream c, suppose that there exists processes W and Z satisfying* (3.2) *and* (3.3)*. Then $\alpha^* = (g')^{-1}(Z)$ is agent's optimal strategy and W is agent's continuation value defined in* (3.1).

Proof. From the construction of H, we have that $\tilde{W}$ is a local supermartingale. Take a localization sequence $\{\tau_n\}$ of its local martingale part. We have

$$W_t \geq \mathbb{E}_t^\alpha\big[e^{-r\tau_n}W_{\tau_n} + r\int_t^{\tau_n} e^{-rt}\big(U(c_t) - g(\alpha_t)\big)dt\big]. \tag{3.4}$$

Sending $\tau_n \to \infty$ on the right-hand side, and using (3.3) and the dominated convergence theorem, we obtain

$$W_t \geq \mathbb{E}_t^\alpha\big[\int_t^\infty e^{-rt}\big(U(c_t) - g(\alpha_t)\big)dt\big].$$

For the strategy α^*, the inequality in (3.4) is an identity. A similar limit argument yields

$$W_t = \mathbb{E}_t^{\alpha^*}\big[\int_t^\infty e^{-rt}\big(U(c_t) - g(\alpha_t^*)\big)dt\big],$$

confirming the optimality of α^*. □

3.3. **Principal's optimization problem.** The key observation by Sannikov is that agent's continuation value can be used as the state variable for principal's problem. Consider the dynamics (3.2), where both Z and c are considered as the control variables for the principal. Define principal's value function as

$$F(W) = \sup_{c,Z,\tau} \mathbb{E}^{\alpha^*}\big[r\int_0^{\tau_0\wedge\tau} e^{-rt}\big(\alpha_t^* - c_t\big)dt \,\big|\, W_0 = W\big],$$

where $\alpha^* = \alpha(Z) = (g')^{-1}(Z)$ and $\tau_0 = \inf\{t \geq 0 : W_t = 0\}$. This is an *optimal control-stopping* problem.

To derive an equation satisfied by F, we use the martingale principle again. To this end, consider

$$F(W_t) = \text{ess sup}_{c,Z,\tau}\mathbb{E}^{\alpha(Z)}\big[r\int_t^\infty e^{-r(s-t)}\big(\alpha(Z_s) - c_s\big)ds\big],$$

and define

$$M_t = r\int_0^t e^{-rs}\big(\alpha(Z_s) - c_s\big)ds + e^{-rt}F(W_t).$$

We expect from the martingale principle that M is a supermartingale under $\mathbb{P}^{\alpha(Z)}$ for arbitrary c, Z, τ, and M is a martingale under $\mathbb{P}^{\alpha(Z)}$ until τ^* for optimal c^*, Z^*, τ^*. Assuming that $F \in C^2(R)$, applying Itô's formula to M, we obtain

$$\begin{aligned} dM_t =& e^{-rt}\big[r(\alpha(Z_t) - c_t) - rF(W_t) + r\big(-\hat{g}(Z_t) + W_t - U(c_t)\big)F'(W_t) + \tfrac{r^2\sigma^2}{2}Z_t^2F''(W_t)\big]dt \\ &+ re^{-rt}\sigma Z_tF'(W_t)dB_t \\ =& re^{-rt}\big[\alpha(Z_t) - c_t - F(W_t) + \big(g(\alpha(Z_t)) + W_t - U(c_t)\big)F'(W_t) + \tfrac{r\sigma^2}{2}Z_t^2F''(W_t)\big]dt \\ &+ re^{-rt}\sigma Z_tF'(W_t)dB_t^{\alpha(Z)}. \end{aligned}$$

The drift must be nonnegative for all controls and zero for optimal controls. Therefore, we obtain the following Hamilton-Jacobi-Bellman (HJB) equation for F:

$$0 = \sup_{c,Z} \Big\{ \alpha(Z) - c - F(W) + \big(g(\alpha(Z)) + W - U(c)\big)F'(W) + \tfrac{r\sigma^2}{2} Z^2 F''(W) \Big\}. \tag{3.5}$$

This HJB equation needs two boundary conditions. On the one hand,

$$F(0) = 0. \tag{3.6}$$

On the other hand, define a retirement function $F_0(U(c)) = -c$. This is the value to the principal if she pays the agent a constant rate c. In this case, the agent no longer works. The other boundary condition is

$$F(\bar{W}) = F_0(\bar{W}) \quad \text{and} \quad F'(\bar{W}) = F_0'(\bar{W}). \tag{3.7}$$

This is a *free-boundary* condition, since $\bar{W}$ is determined together with the solution F. The condition on the first order derivative is called the *smooth pasting* condition.

Now we are ready to state the main theorem.

Theorem 3.2. *Suppose that the HJB equation* (3.5) *together with its boundary conditions* (3.6) *and* (3.7) *admit a solution $F \in C^2(0, \bar{W})$ and $\bar{W} \in [R, \infty)$. Let $Z(W)$ maximizes*

$$\alpha(Z) + g(\alpha(Z))F'(W) + \tfrac{r\sigma^2}{2} Z^2 F''(W),$$

and $c(W)$ maximizes

$$-c - U(c)F'(W).$$

Choose $W^* = \max_{W \in [R, \bar{W}]} F(W)$ and define W via the following SDE

$$dW_t = r\big(-\hat{g}(Z(W_t) + W_t - U(c(W_t)))\big)dt + r\sigma Z(W_t)dB_t, \quad W_0 = W^*.$$

Then principal's optimal contract is given by $c(W)$, agent's optimal effort is $\alpha(Z(W))$, $\tau_0 = \inf\{t \geq 0 : W_t = 0\}$, and $\tau_e = \inf\{t \geq 0 : W_t = \bar{W}\}$.

Remark 3.3. The proof is a verification argument, see [Sannikov, 2008] and [Strulovici and Szydlowski, 2015]. In particular, in order to ensure (3.5) admits a C^2 solution, one needs to restrict $Z \geq \underline{Z}$ for some positive constant $\underline{Z}$. This ensures (3.5) is uniformly elliptic, but restricts principal's admissible contracts. An numeric example is given in Figure 1.

4. DeMarzo and Sannikov (2006)

4.1. **Problem formulation.** We present [DeMarzo and Sannikov, 2006] in this section.

The agent, if hired, produces an output process X which follows

$$dX_t = (\mu - \alpha_t)dt + dB_t^\alpha,$$

where μ is a constant and $dB_t^\alpha = dB_t + \alpha_t dt$. The agent's effort α is a "shirking" action, which can be interpreted as inefficient implementation or working less hard than he should be. This shirking action yields private benefit to the agent. The agent can choose any non-negative α. The benefit is described by a parameter $\lambda \in [0,1]$. The larger λ is, the higher benefit agent gets. The cumulative compensation paid by the principal is described by a non-decreasing process C. The agent is risk neutral and has a reservation utility R. Given C, Agent's optimization problem is

$$\mathbb{E}^\alpha\Big[\int_0^\infty e^{-\gamma t}\big(dC_t + \lambda\alpha_t dt\big)\Big] \to \text{Max.}$$

Let α^* be agent's optimal strategy for the problem above.

The principal pays the agent a non-decreasing process C. The agent has limited liability. When his continuation value reaches zero, the project is liquidated, the principal gets L and the agent gets nothing more and stops shirking. Principal's optimization problem is

$$\mathbb{E}^{\alpha^*}\Big[\int_0^\tau e^{-rt}[(\mu-\alpha_t)dt - dC_t] + e^{-r\tau}L\Big],$$

subject to agent's participation constraint.

4.2. **Agent's optimization problem.** Define agent's continuation value

$$W_t = \text{ess sup}_\alpha \mathbb{E}_t^\alpha\Big[\int_t^\infty e^{-\gamma(s-t)}\big(dC_s + \lambda\alpha_s ds\big)\Big].$$

Using the martingale principal similarly as before, we obtain the dynamics of W as

$$dW_t = \big[\gamma W_t + \inf_{\alpha\geq 0}\{\alpha_t(Z_t-\lambda)\}\big]dt - dC_t + Z_t dB_t. \tag{4.1}$$

When $Z \geq \lambda$, the optimizer $\alpha^* = 0$ and $\inf_{\alpha\geq 0}\{\alpha_t(Z_t-\lambda)\} = 0$. If $Z < \lambda$ has positive measure under $dt \times d\mathbb{P}$, the drift of (4.1) is $-\infty$. Therefore, wellposedness for agent's problem impose a constraint for the contract sensitivity

$$Z \geq \lambda \quad dt \times d\mathbb{P} \text{ a.s.} \tag{4.2}$$

We also need the transversality condition

$$\lim_{\tau\to\infty} \mathbb{E}^\alpha\big[e^{-r\tau}W_\tau\big] = 0, \quad \text{for strategy } \alpha \text{ and any stopping time } \tau\to\infty. \tag{4.3}$$

We constrain the compensation process C and agent's strategy α to the classes which satisfy the following integrability property.

$$\mathbb{E}^\alpha\Big[\int_0^\infty e^{-rt}U(c_t)dt\Big] < \infty \quad \text{and} \quad \mathbb{E}^\alpha\Big[\int_0^\infty e^{-rt}g(\alpha_t)dt\Big] < \infty.$$

An argument similar to Proposition 3.1 yields

Proposition 4.1. *Let W and Z be two processes satisfying*

$$dW_t = \gamma W_t dt - dC_t + Z_t dB_t,$$

the constraint (4.2) *and the transversality condition* (4.3). *Then $\alpha^* = 0$ is agent's optimal strategy and W is agent's continuation value.*

4.3. **Principal's optimization problem.** Consider W as the state variable for principal's optimization problem. Define principal's value function as

$$F(W_t) = \text{ess sup}_{Z\geq\lambda}\mathbb{E}_t\big[\int_t^\tau e^{-r(s-t)}[\mu ds - dC_s] + e^{-r(\tau-t)}L\big],$$

where $\tau = \inf\{s \geq t : W_s \leq 0\}$. This is called a *singular control* problem.

Applying the martingale principal to principal's optimization problem, assuming that C is differentiable, we can get the following HJB equation

$$rF(W) = \max_{Z\geq\lambda, C}\big\{\mu + \gamma W F'(W) - (1 + F'(W))C' + \tfrac{Z^2}{2}F''(W)\big\}. \tag{4.4}$$

However, C' can be arbitrarily large, in order to keep the HJB equation well-posed, we need

$$F'(W) \geq -1.$$

This inequality can also be understood from an economics point of view. The principal can always pay the agent $\Delta C > 0$. In this case, agent's continuation value decreases by ΔC

$$F(W) \geq F(W - \Delta C) - \Delta C,$$

which implies $F'(W) \geq -1$. On can also see from (4.4) that $C' = 0$ whenever $F'(W) > -1$. Therefore, the principal delay the payment to the agent until $F'(W) = -1$.

The boundary conditions are

$$F(0) = L, \quad F'(\bar{W}) = -1, \quad \text{and} \quad F''(\bar{W}) = 0.$$

The third condition is the smooth pasting condition. The boundary conditions at $\bar{W}$ imply

$$rF(\bar{W}) + \gamma\bar{W} = \mu.$$

That is, the payments are postponed until the project's expected return is used up by the sum of the individual expected returns.

Theorem 4.2. *Suppose that $\gamma > r$. Consider the ODE system :*

$$\mu + \gamma W F'(W) + \frac{1}{2}\lambda^2 F''(W) - rF(W) = 0, \quad F'(W) \geq -1, \quad W \in [0, \bar{W}),$$

$$F'(W) = -1 \quad W \geq \bar{W},$$

$$F(0) = L, \quad F''(\bar{W}) = 0.$$

Assume that it admits a concave solution $F \in C^2$ and $\bar{W} < \infty$. Then, *(i) F is Principal's value function.*

(ii) When $W \in [0, \bar{W}]$, truth-telling is optimal, i.e., $a^* \equiv 0$. Moreover, it is optimal to set $Z \equiv \lambda$, and the payments C to be the reflection process which keeps W_t within $[0, \bar{W}]$. That is, C is the smallest increasing process such that

$$W_t = W_0 + \int_0^t \gamma W_s ds - C_t + \lambda B_t$$

stays within $[0, \bar{W}]$. In particular, when $W_t \in (0, \bar{W})$, $dC_t = 0$. The contract terminates once W hits 0.

(iii) When $W > \bar{W}$, then the optimal contract pays an immediate payment of $W - \bar{W}$ to the agent, and the contract continues with the agent's new initial utility $\bar{W}$.

Proof. Let F denote the solution to the ODE system and $\hat{F}$ denote the value function. We first show that $\hat{F} \le F$. To see that, introduce

$$G_t := -\Delta C_0 + \int_0^t e^{-rs}[(\mu - \alpha_s)ds - dC_s] + e^{-rt} F(W_t).$$

By Itô's formula we have

$$\begin{aligned} e^{rt} dG_t =& \Big[\mu - \alpha_t[1 + Z_t F'(W_t)] + \gamma W_t F'(W_t) + \tfrac{1}{2} Z_t^2 F''(W_t) - rF(W_t)\Big] dt \\ & - [1 + F'(W_{t-})] dC_t + Z_t F'(W_t) dB_t^\alpha. \end{aligned}$$

Since $F'(W) \ge -1$ for $W \in [0, \bar{W}]$, we have

$$1 + lF'(W_t) \ge 0, 1 + F'(W_t) \ge 0.$$

This, together with the assumptions that F is concave and $Z \ge \lambda$, implies that

$$e^{rt} dG_t \le \Big[\mu + \gamma W_t F'(W_t) + \frac{1}{2}\lambda^2 F''(W_t) - rF(W_t)\Big] dt + Z_t F'(W_t) dB_t^\alpha.$$

When $W_t \in [0, \bar{W}]$, the drift is zero. When $W_t > \bar{W}$, by $rF(\bar{W}) + \gamma \bar{W} = \mu$ and other boundary conditions, we can compute that the drift is

$$[r - \gamma][W_t - \bar{W}] < 0,$$

thanks to $r < \gamma$. Therefore, in both the cases we have

$$e^{rt} dG_t \le Z_t F'(W_t) dB_t^\alpha.$$

Thus, G is a $\mathbb{P}^\alpha$-supermartingale. Notice that $F(W) = G_0$ and $F(W_\tau) = F(0) = L$. We have then

$$F(W) = G_0 \ge \mathbb{E}^\alpha[G_\tau].$$

for all a, τ. Taking the supremum, we get $F = G_0 \ge \hat{F}$.

For the strategy in the theorem, G is a martingale.

$$e^{rt} dG_t = Z_t F'(W_t) dB_t.$$

Then, we have

$$F(W) = G_0 = \mathbb{E}[G_\tau].$$

Thus, the upper bound is attained, and $\hat{F}(W) = F(W)$ for $W \in [0, \bar{W}]$.

Finally, for $W > \bar{W}$, note that, by the "immediate payment" argument above, the value function satisfies, for $W > W' > \bar{W}$,

$$\hat{F}(W) \geq \hat{F}(\bar{W}) - W + \bar{W}$$

Since we have $\hat{F}(\bar{W}) = F(\bar{W}) = F(W) - \bar{W} + W$, we have $\hat{F}(W) \geq F(W)$. □

4.4. **Implementation Using Standard Securities.** We now want to show that the above contract can be implemented using real-world securities, namely, equity, long-term debt and credit line. The implementation will be accomplished using the following:

- The firm starts with initial capital K and possibly an additional amount needed for initial dividends or cash reserves.

- The firm has access to a credit line up to a limit of C^L. The interest rate on the credit line balance is γ. The agent decides on borrowing money from the credit line and on repayments to the credit line. If the limit C^L is reached, the firm/project is terminated.

- Shareholders receive dividends which are paid from cash reserves or the credit line, at the discretion of the agent.

- The firm issues a long (infinite) term debt with continuous coupons paying at rate x. If the firm cannot pay a coupon payment, the project is terminated.

The agent will be paid by a fraction of dividends. We assume that once the project is terminated the agent does not receive anything from his holdings of equity. Here is the result that shows precisely how the optimal contract is implemented.

Theorem 4.3. *Suppose that the credit line has interest rate γ, and that the long-term debt satisfies*

$$x = \mu - \gamma C^L. \tag{4.5}$$

Assume the dividends are paid only at the times the credit line balance hits zero, making the credit line balance the process that reflects at zero. If the agent is paid by a proportion λ of the firm's dividends, he will not misreport the cash flows, and will use them to pay the debt coupons and the credit line before issuing dividends. Denoting the current balance of the credit line by M_t, the agent's expected utility process satisfies

$$W_t = \lambda(C^L - M_t). \tag{4.6}$$

If in addition

$$C^L = \bar{W}/\lambda \tag{4.7}$$

then the above capital structure of the firm implements the optimal contract.

Proof. Denote by δ_t the cumulative dividends process. By that, we mean that $d\delta_t$ is equal to whatever money is left after paying the interest $\gamma M_t dt$ on the credit line and the debt coupons xdt. Since the total amount of available funds is equal to the balance of the credit line M plus the reported profit X, and since $M+X$ is divided between the credit line interest payments, debt coupon payments and dividends, we have

$$dM_t = \gamma M_t dt + xdt + d\delta_t - dX_t$$

With W_t as in (4.6), and from (4.5), we have

$$dW_t = -\lambda dM_t = \gamma W_t dt - \lambda d\delta_t + \lambda dB_t$$

If we set $dC_t = \lambda d\delta_t$, then this corresponds to the agent's utility with zero savings and $Z \equiv \lambda$, which implies that the agent will not have incentives to misreport. Moreover, since the dividends are paid when $M_t = 0$, which, by (4.7) is equivalent to $W_t = \bar{W}$, we see that the optimal strategy is implemented by this capital structure. □

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Figure 1. Sannikov (2008)

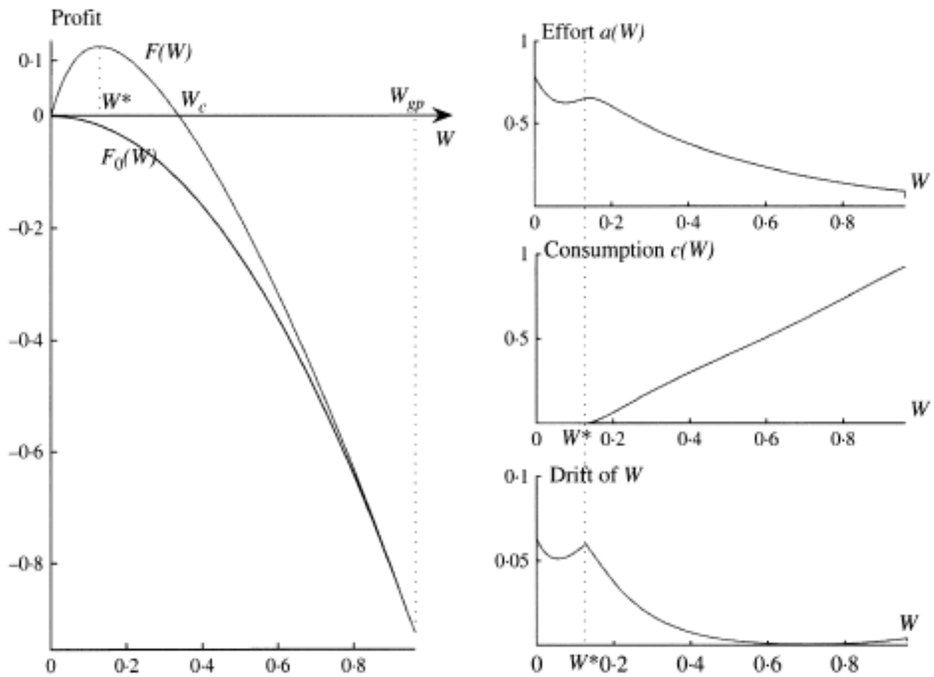

FIGURE 1

Function F for $u(c) = \sqrt{c}$, $h(a) = 0{\cdot}5a^2 + 0.4a$, $r = 0{\cdot}1$ and $\sigma = 1$. Point W^* is the maximum of F

DeMarzo and Sannikov (2006)

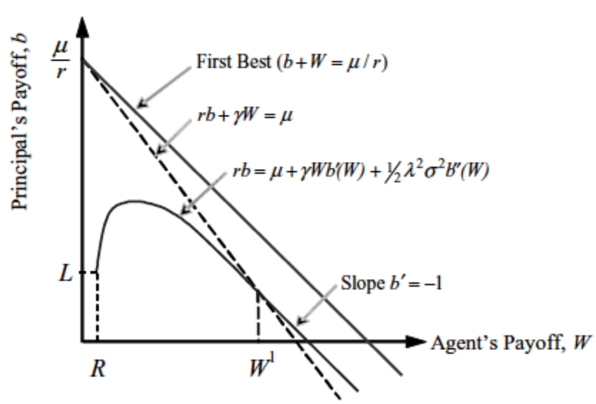

Figure 1. The principal's value function $b(W)$. The principal's value function starts at (R, L), and obeys the differential equation (15) until the point W^1, and then continues with slope -1.